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DISCRETE MATHEMATICS

On colored designs — I

Y. Caro[a,*], Y. Roditty[b,c], J. Schönheim[b]

[a] *Department of Mathematics, School of Education, University of Haifa-Oranim, Tivon, Israel 36910*
[b] *School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Tel-Aviv, Israel 69978*
[c] *Department of Mathematics, Beit-Berl College, Kfar-Saba, Israel 44905*

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Abstract

A colored version of the H-design concept is studied. The problem of whether or not the existence of an H-design for a graph G implies the existence of the corresponding *colored designs* is solved for small graphs H in the cases of two and three colors.

1. Introduction, definitions and notation

Graphs in this article are loopless. For basic definitions we refer to [6]. In this article we shall deal with the concept of *colored designs*, the *Colored Design Problem*, and supply solutions for many cases.

However, before defining the colored design concept we need to mention some definitions of known concepts and introduce some new ones.

A graph G is said to have an *H-decomposition*, denoted $H|G$, if the edge-set of G is the disjoint union of the edge-sets of isomorphic copies of H. The set of the above copies of H is called an *H-design* and generalizes the block design concept. Block designs are H-designs where H and G are complete graphs.

Several surveys on H-designs have been written but we shall mention only the recent book of Bosak [3].

Graphs having no multiple edges will be denoted $G, H, \ldots$, and G^λ denotes the multigraph having the same vertex set as G and obtained by replacing each edge of G by an edge of multiplicity λ.

Let $L = \{1, 2, \ldots, \lambda\}$ be a set of λ colors. Denote by CG^λ the edge colored graph G^λ in which all λ colors are used on every multiple edge. In other words, the graph CG^λ is the union of λ monochromatic copies of G, any two copies having distinct colors. Suppose H is a graph with $e(H)$ edges, such that $\lambda | e(H)$. A *λ-coloring ϕ of*

* Corresponding author.

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H, is a mapping $\phi : E(H) \rightarrow \{1,2,\ldots,\lambda\}$ such that the number of edges colored k is the same, namely, $e(H)/\lambda$, for every $k \in L$. These colorings have been called *uniform* (see [5]). The graph H colored by a specific ϕ will be denoted $C_\phi H$. The λ-colored graphs $C_{\phi_1}H_1$ and $C_{\phi_2}H_2$ are called *isochromatic* if H_1 and H_2 are isomorphic and there is an isomorphism $f : H_1 \rightarrow H_2$ such that for every edge $e \in E(H_1)$ the color of e is preserved by f, i.e. $\phi_1(e) = \phi_2(f(e))$. For this we shall use the notation $C_{\phi_1}H_1 \sim C_{\phi_2}H_2$.

Now we can define the colored design concept.

Definition 1. The graph CG^λ is said to have a *$C_\phi H$-decomposition*, denoted $C_\phi H|CG^\lambda$ if the colored edge-set of CG^λ is the disjoint union of the colored edge set of isochromatic copies of $C_\phi H$. The set of those copies is called also *a colored design*.

If for a fixed graph H the decomposition $C_\phi H|CG^\lambda$ exists for every λ-coloring ϕ then we write $CH|CG^\lambda$.

In the following, we define for $\lambda = 2$ and $\lambda = 3$ the sets of colors to be $L = \{A,B\}$ and $L = \{A,B,C\}$, respectively.

Notation. The vertex set of K_n is defined to be Z_n, with addition of vertex labels done mod n for n odd, and $\{Z_{n-1} \cup \infty\}$ when n is even, where addition of vertex labels is done $\bmod(n-1)$. By $K_m(t)$ we denote the complete t-partite graph in which each part is of size m, and $K_{m,n}$ is the complete bipartite graph whose two parts are of sizes m,n.

The following example illustrates the concept of Definition 1.

Example. $CP_3|CK_4^2$ holds, as shown by the decomposition $[(1,2,3),(1,\infty,2)] \pmod 3$ with the first edge in the path (x,y,z) colored A. Clearly, P_3 has only one 2-coloring.

In particular, we give the following:

Definition 2. A λ-coloring ϕ of a given graph H is called *permutable* if for every λ-coloring ϕ' of H obtained by permuting the colors in $C_\phi H$ we have $C_\phi H \sim C_{\phi'}H$.

In particular for $\lambda = 2$ the coloring is permutable if for every edge $e \in E(H)$,

$$\phi'(e) = \begin{cases} A \text{ when } \phi(e) = B\,, \\ B \text{ when } \phi(e) = A \end{cases} \tag{1}$$

implies, $C_\phi H \sim C_{\phi'}H$. In this case we say symmetric instead of permutable.

Notice that there are graphs H such that every λ-coloring of H is permutable. For instance, $K_{1,\lambda t}, \lambda t K_2$. We call such graphs λ *color symmetric.*

A simple theorem but of great importance is the following:

Theorem 1.1. *If ϕ is a permutable λ-coloring of H and $H|G$ then, $C_\phi H|CG^\lambda$.*

Corollary 1.2. *If H is λ color symmetric then, $H|G \to CH|CG^\lambda$.*

2. The colored design problems and main results

Clearly, the colored design given by a decomposition $C_\phi H|CG^\lambda$ can exist only if the corresponding H-design given by $H|G^\lambda$ exists. The main question here is:

The colored design problem: *Is this necessary condition also sufficient ? or is it even sufficient for every ϕ, in other words: does $H|G^\lambda$ imply $CH|CG^\lambda$?.*

As mentioned by Colbourn and Stinson in [5], this question has been considered by Wilson in the case of block designs and the condition seems to be asymptotically sufficient.

Of course, asymptotic results do no discourage combinatorists to look for exact results. In this direction, we mention here that in the above-cited paper of Colbourn and Stinson the colored block design problem is solved for $H \in \{K_3, K_4\}$ including colorings called *non-uniform*, where the color classes of CH have different sizes. The few noncovered cases by this paper have been settled since by Bennett et al. (see [1]).

Another particular case of the colored design problem is the following one posed recently by Yu (see [7]):

Given a path P_{2k+1} with its edges colored red and blue so that there are k edges of each color, given also K_n^2 colored red and blue so that there is a red and blue copy of K_n. If $n(n-1) \equiv 0 \pmod{2k}$, is there a decomposition of the colored K_n^2 into copies of the colored path ?

Here $\lambda = 2$, $H = P_{2k+1}$ and ϕ is any 2-coloring of H.

Understandably, we shall restrict our investigation concerning the existence of colored designs to such H, G and λ for which necessary and sufficient conditions for $H|G^\lambda$ are known. For the graphs which are our concern in this paper, one may find the required decompositions either in [3] or as an easy result follows from the colored case.

For $\lambda = 1$ the problem reduces to the uncolored case. We consider $\lambda \in \{2,3\}$ and concentrate on the cases $G = K_n$, and $G = K_{m,n}$.

The cases for which a complete answer is given to the Colored Design Problem are formulated in the following Main Theorems.

Theorem 2.1. *The necessary condition for $CH|CK_n^2$,*

$$H|K_n^2 \tag{2}$$

is also sufficient for all 13 *graphs having two or four edges, except for the case* $H = C_4, n = 5$, *where in spite of* $C_4|K_5^2$ *there is no colored decomposition for the coloring ABAB.*

Notice that to have (2) one has to have $e(H)|n(n-1)$, and this condition is always satisfied for $e(H) = 2$ and gives $n \equiv 0, 1 \pmod 4$ for $e(H) = 4$.

Theorem 2.2. *The necessary condition for* $CH|CK_n^3$,

$$H|K_n^3 \tag{3}$$

is also sufficient for all 5 *graphs having three edges.*

Again notice that a necessary condition for (3) is

$$e(H)|3n(n-1)/2.$$

3. Methods used in the proofs

We shall use both the recursive composition method establishing the so-called building blocks and the method of differences called base blocks, known from H-Design Theory, and get colored designs. The crux of the problem is whether the base blocks and building blocks can be colored in a way to lead to a colored design.

For most of the 13 graphs from Theorem 2.1 we shall determine the colored building blocks, giving the colored decomposition of CK_n^2 for small values of n (satisfying the necessary condition), which enables us to start the inductive proof. Then also the colored decompositions of $CK_{a,b}^2$ for some small a, b are established (in some cases additional decompositions of certain small graphs are also built) in order to accomplish the induction. For the remaining graphs the proof is direct by determining the colored base blocks by the method of differences emphasizing the coloring. Some building blocks are also obtained in this way. An example will make clear how the coloring of the base blocks has to be chosen.

Example. It is known that $P_{2k+1}|K_{2k+1}^2$ (see [7,11]). One way to see this is to label the vertices of K_{2k+1} by the members of the additive group, Z_{2k+1}. The paths of the decompositions are:

$$(2k, 2k+1, 2k-1, 1, 2k-2, 2, 2k-3, 3, \ldots, k-1, k) \pmod{2k+1},$$

since the differences between consecutive vertices are $D = \{1, -2, 3, -4, \ldots, -4, 3, -2, 1\}$ so that each absolute value from 1 to k occurs exactly twice. One can get a 2-coloring ϕ and the corresponding colored design defined by using color A to k edges corresponding to any choice of k different absolute values from D. For such ϕ we have $C_\phi P_{2k+1} \mid CK^2_{2k+1}$.

The next section includes the construction of colored building blocks followed by induction proofs and the construction of colored base blocks.

We shall emphasize that ending each proof an argument as above has to be used to ensure that every edge in G occurs in each color exactly once.

4. Proof of main theorems

Before proving the main theorems we give some notation and preparatory results. We shall use the notation: $V(K_{2,3}) = \{a,b\} \cup \{0,1,2\}$, $V(K_{2,4}) = \{a,b\} \cup \{0,1,2,3\}$, $V(K_{3,4}) = \{a,b,c\} \cup \{0,1,2,3\}$.

4.1. Preliminary results concerning $\lambda = 2$

4.1.1. Building blocks for P_5

There are exactly three possible 2-colorings of P_5 (the path on 5 vertices and 4 edges). Two symmetrical colorings ϕ_1 : *ABAB* and ϕ_2 : *AABB*, and the coloring ϕ_3 : *ABBA*.

The additional colorings *BABA*, *BBAA* and *BAAB* differ only by notation from the first three.

Lemma 4.1. *$CP_5 | CG^2$ for $G \in \{K_{2,4}, K_{3,4}\}$.*

Proof. It is well known (see [8]) that for $i = 2,3$ $P_5 | K_{i,4}$. Therefore, by Corollary 1.2, $C_\phi P_5 | CK^2_{i,4}$, for $\phi \in \{\phi_1, \phi_2\}$. For ϕ_3 the required decomposition of $K^2_{2,3}$ and $K^2_{2,4}$ is shown below and using the fact that $K_{3,2} | K_{3,4}$ the decomposition of $K_{3,4}$ follows:

$$CK^2_{2,3} \quad (0,a,1,b,2)\ (1,a,2,b,0),$$
$$(1,b,0,a,2),$$

$$CK^2_{2,4} \quad (0,a,1,b,3)\ (0,b,3,a,2),$$
$$(1,a,0,b,2)\ (1,b,2,a,3).$$

Corollary 4.2. *For $i = 2,3$ $CP_5 | CK^2_{it,4s}$, $t \geqslant 1$, $s \geqslant 1$ (integers).*

Lemma 4.3. *$CP_5 | CK^2_i$ for $i \in \{5,8,9,12,13\}$.*

Proof. One has $P_5 | K^2_5$, by any of the decompositions:

(i) $(0,2,3,4,1) \pmod 5$,

(ii) $(0,1,3,4,2) \pmod 5$.

The sequence of differences is $(2,1,1,2)$ in (i) and $(1,2,1,-2)$ in (ii). In order to obtain a 2-coloring choose A in two places occupied by different absolute values, and get ϕ_2 or ϕ_1 in (i), and ϕ_2 or ϕ_3 in (ii). This gives all possible colorings.

The colored decomposition of K_8^2 is:

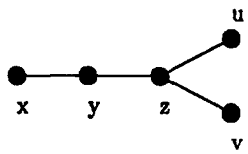
For both $\phi_1 : ABAB$ and $\phi_2 : AABB$: $\{(\infty, 1, 2, 3, 5), (5, 0, 4, 1, \infty)\} \pmod 7$.

$\phi_3 : ABBA$: $\{(0, 3, 1, \infty, 2), (1, 2, 5, 6, 4)\} \pmod 7$.

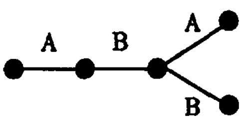
For K_9^2 observe that $K_9 = K_5 \cup K_5 \cup K_{4,4}$ where the two graphs K_5 share one common vertex and apply Lemma 4.1 and use the result on K_5. The decompositions of K_{12}^2 and K_{13}^2 are:

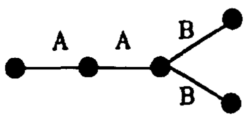
	ABBA	**AABB**	**ABAB**
CK_{12}^2	(0, 1, 4, 5, 8)	(0, 1, 3, 4, 6)	(0, 1, 3, 5, 6)
	(0, 2, 6, 8, 1)	(0, 3, 7, 1, ∞)	(0, 3, 7, 1, ∞)(mod 11)
	(2, ∞, 1, 6, 0)	(∞, 0, 5, 8, 1)	(∞, 0, 3, 7, 1)
CK_{13}^2	(0, 1, 3, 4, 6)	(0, 1, 3, 4, 6)	(0, 1, 3, 5, 6)
	(0, 3, 7, 10, 1)	(0, 3, 7, 10, 1)	(0, 3, 7, 11, 1)(mod 13)
	(0, 5, 11, 3, 9)	(0, 5, 11, 3, 9)	(0, 5, 11, 4, 9)

4.1.2. Building blocks for H_5 =



For H_5 drawn above (denoted $(x, y, z; u, v)$) there are two 2-colorings, namely,

ϕ_1 : 

ϕ_2 : 

Lemma 4.4. *$CH_5 | CG^2$ where, $G \in \{K_{2,4}, K_{3,4}, K_5, K_8, K_9, K_{12}, K_{13}\}$.*

Proof. Both colorings can be applied on the following decomposition:

$CK_{2,4}^2 = (b, 1, a; 3, 0), (a, 0, b; 2, 1), (b, 3, a; 1, 2), (a, 2, b; 0, 3)$.

For the other graphs the decomposition for ϕ_1 and ϕ_2 are:

	ABAB	**AABB**
$CK^2_{3,4}$	$(b, 0, a; 2, 1)$	$(a, 0, b; 2, 3)$
	$(c, 1, b; 2, 0)$	$(c, 2, a; 1, 3)$
	$(a, 3, b; 1, 2)$	$(b, 1, c; 0, 3)$
	$(b, 3, c; 0, 2)$	$(3, a, 1; b, c)$
	$(a, 0, c; 2, 1)$	$(3, c, 0; a, b)$
	$(c, 3, a; 1, 2)$	$(3, b, 2; a, c)$
CK^2_5	$(0, 4, 1; 3, 2) \pmod 5$	the same
CK^2_8	$(0, 1, 3; 6, \infty) \pmod 7$	the same
	$(\infty, 5, 2; 0, 1) \pmod 7$	
CK^2_9	$(0, 1, 3; 5, 4) \pmod 9$	the same
	$(0, 3, 7; 2, 1) \pmod 9$	
CK^2_{12}	$(0, 1, 3; 5, 4) \pmod{11}$	the same
	$(0, 3, 7; 1, \infty) \pmod{11}$	
	$(\infty, 0, 5; 9, 8) \pmod{11}$	
CK^2_{13}	$(0, 1, 3; 5, 4) \pmod{13}$	the same
	$(0, 3, 7; 11, 10) \pmod{13}$	
	$(0, 5, 11; 4, 3) \pmod{13}$	

Corollary 4.5. *$CH_5 | CK^2_{is,4r}$, for $i \in \{2, 3\}$, $r \geqslant 1$, $s \geqslant 1$ (integers).*

4.1.3. Building blocks for C_4

Clearly there are only two possible 2-colorings of C_4, namely,

$$\phi_1 : ABAB \quad \text{and} \quad \phi_2 : AABB.$$

Lemma 4.6. (i) *$CC_4 | CG^2$ for $G \in \{K_4, K_{2,3}, K_{2,4}, K_9\}$.*
(ii) *$C_{\phi_2}C_4 | CK^2_5$.*

Proof. The required colored designs are:

	ABAB	**AABB**
CK^2_4	$(\infty, 0, 1, 2) \pmod 3$	the same
CK^2_5	does not exist	$(2, 1, 3, 4) \pmod 5$

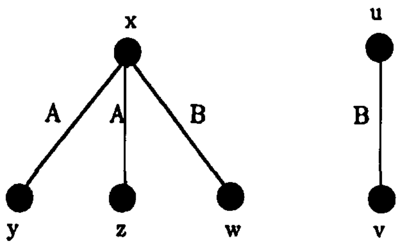
$CK^2_{2,3}$	$(a, 1, b, 0)$	the same
	$(a, 0, b, 2)$	the same
	$(a, 2, b, 1)$	the same
$CK^2_{2,4}$	$(a, 0, b, 1)$	the same
	$(a, 1, b, 2)$	the same
	$(a, 2, b, 3)$	the same
	$(a, 3, b, 0)$	the same
CK^2_9	$(0, 1, 3, 8)\,(\mathrm{mod}\,9)$	the same
	$(0, 3, 8, 6)\,(\mathrm{mod}\,9)$	the same

The above decomposition of CK^2_5 does not exist for ϕ_1 since there is only one decomposition $C_4|K^2_5$ which is not compatible with ϕ_1.

Corollary 4.7. *$CC_4|CG^2$ for $G \in \{K_{2a,3b}, K_{2a,4b}\}$.*

4.1.4. Building blocks for $H = K_{1,3} \cup K_2$

There is only one 2-coloring of H, namely,



In the decomposition below this coloring shall be applied to the notation $H = [(x; y, z, w)(u, v)]$.

Lemma 4.8. *Let H be defined as above then, $CH|CG^2$ for*

$$G \in \{K_8, K_{2,4}, K_9, K_{3,4}, K_{12}, K_{13}\}.$$

Proof. The required colored designs are:

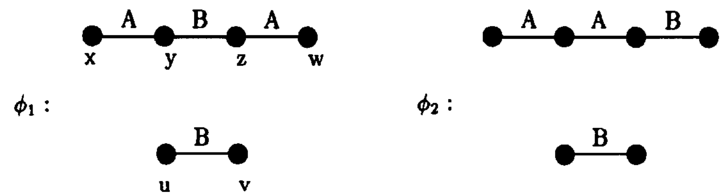
$CK^2_{2,4}$	$[(a; 0, 1, 2)(b, 3)]$	$[(a; 3, 2, 0)(b, 1)]$
	$[(b; 3, 2, 0)(a, 3)]$	$[(b; 0, 2, 3)(a, 1)]$
$CK^2_{3,4}$	$[(a; 0, 1, 2)(c, 3)]$	$[(a; 3, 2, 0)(b, 1)]$
	$[(b; 0, 1, 2)(a, 3)]$	$[(b; 3, 2, 0)(c, 1)]$
	$[(c; 0, 1, 2)(b, 3)]$	$[(c; 3, 2, 0)(a, 1)]$
CK^2_8	$[(0; 1, 2, \infty)(3, 6)]\,(\mathrm{mod}\,7)$	$[(0; 3, \infty, 6)(4, 2)]\,(\mathrm{mod}\,7)$

CK_9^2	$[(1;2,3,5)(6,0)]\ (\mathrm{mod}\ 9)$	$[(1;7,5,2)(6,8)]\ (\mathrm{mod}\ 9)$
CK_{12}	$[(1;2,4,5)(6,8)]\ (\mathrm{mod}\ 11)$ $[(1;5,6,2)(3,\infty)]\ (\mathrm{mod}\ 11)$ $[(1;\infty,3,4)(2,7)]\ (\mathrm{mod}\ 11)$	
CK_{13}	$[(1;2,3,4)(5,7)]\ (\mathrm{mod}\ 13)$ $[(1;5,6,2)(3,7)]\ (\mathrm{mod}\ 13)$ $[(1;4,7,6)(2,8)]\ (\mathrm{mod}\ 13)$	

Corollary 4.9. *$CH|G^2$ for $G \in \{K_{2a,3b}, K_{2a,4b}\}$.*

4.1.5. Building blocks for $P_4 \cup K_2$

For $H = P_4 \cup K_2$ we have two possible colorings, namely,



We denote this graph by $[(x, y, z, w)(u, v)]$ where the colors on the edges are placed according to that order and to the various colorations.

Lemma 4.10. *$CH|CG^2$ for $G \in \{K_8, K_9, K_{12}, K_{13}, K_{3,4}, K_{4,4}\}$.*

Proof. The required colored designs are:
For $K_{3,4}^2$ and for both colorings:

$$[(0,a,1,b)(c,2)], [(2,b,3,c)(a,0)], [(0,c,1,a)(b,2)],$$

$$[(2,a,3,b)(c,0)], [(0,b,1,c)(a,2)], [(2,c,3,a)(b,0)].$$

For $K_{4,4}^2$ observe that $2K_{2,2}|K_{4,4}$, and it is easy to see that $C(P_4 \cup K_2)|C(2K_{2,2}^2)$ for both colorings, Hence, we are done.

For K_8^2, we have

$$\{[(\infty,1,0,2)(3,6)], [(1,2,0,3)(\infty,6)]\}\ (\mathrm{mod}\ 7)$$

for the coloring $\phi_1 : ABA, B$, and

$$\{[(\infty,1,0,2)(3,6)], [(1,3,6,0)(\infty,2)]\}\ (\mathrm{mod}\ 7)$$

for the coloring $\phi_2 : AAB, B$.

For K_9^2 we have for both colorings the following decomposition:

$$\{[(8,0,7,1)(6,2)], [(6,2,5,3)(0,1)]\}\ (\mathrm{mod}\ 9).$$

For K_{12}^2, we have

$$\{[(1,2,5,10)(\infty,6)],[(\infty,1,6,8)(7,9)],[(1,4,5,9)(2,6)]\}\ (\mathrm{mod}\ 11),$$

for ϕ_1 and

$$\{[(0,5,8,9)(\infty,4)],[(\infty,0,2,7)(1,3)],[(0,4,5,8)(2,6)]\}\ (\mathrm{mod}\ 11),$$

for ϕ_2.

Finally, for K_{13}^2 the decomposition

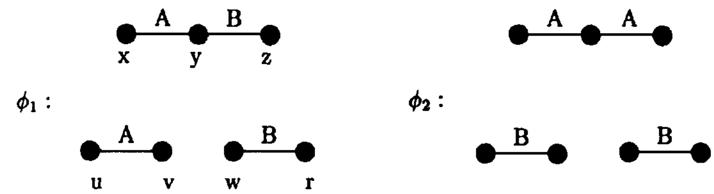
$$\{[(1,3,4,7)(6,8)],[(1,5,10,11)(2,6)],[(1,7,10,2)(3,9)]\}\ (\mathrm{mod}\ 13),$$

is valid for both colorings.

Corollary 4.11. $CH|K_{3a,4b}^2; K_{4c,4d}^2$.

4.1.6. Building blocks for $H = P_3 \cup 2K_2$

For the graph H there are two possible colorings, namely,



Observe that ϕ_1 is symmetric while ϕ_2 is not. We shall use the notation $[(x,y,z)(u,v)(w,r)]$ and apply the appropriate coloring.

Lemma 4.12. *$CH|CG^2$ for $G \in \{K_{3,4}, K_{4,4}, K_8, K_9, K_{12}, K_{13}\}$.*

Proof. For the color ϕ_1, by Corollary 1.2 it is enough to show that $H|G$ for the above defined graphs G. The decomposition of $K_{3,4}$ is

$$[(0,a,1)(b,2)(c,3)],[(0,b,1)(a,3)(c,2)],[(0,c,1)(a,2)(b,3)].$$

For $K_{4,4}$ observe that $H|2K_{2,2}$ and $2K_{2,2}|K_{4,4}$. Thus, we are done. For K_8 and K_9, respectively, the required decompositions are:

$$[(0,1,3)(2,5)(4,\infty)]\ (\mathrm{mod}\ 7),[(0,1,3)(2,5)(4,8)]\ (\mathrm{mod}\ 9).$$

For the coloring ϕ_2 one can apply the coloring to the decompositions:

$$CK_{3,4}^2 = [(0,c,1)(a,2)(b,3)],[(0,a,1)(b,2)(c,3)],[(2,a,3)(b,1)(c,0)],$$
$$[(0,b,1)(a,3)(c,2)],[(2,b,3)(a,0)(c,1)],[(2,c,3)(a,1)(b,0)].$$

$$CK_8^2 = \{[(0,1,3)(2,5)(4,\infty)], [(0,3,\infty)(1,2)(4,6)]\} \pmod 7.$$

$$CK_9^2 = \{[(0,1,3)(2,5)(4,8)], [(0,3,7)(2,3)(4,6)]\} \pmod 9.$$

For K_{12} and K_{13}, one has

	ϕ_2	ϕ_1
CK_{12}^2	$[(0,1,4)(2,7)(\infty,3)] \pmod{11}$	$[(1,2,5)(6,0)(\infty,3)] \pmod{11}$
	$[(\infty,0,5)(1,3)(4,7)] \pmod{11}$	$[(1,3,8)(\infty,2)(7,9)] \pmod{11}$
	$[(0,2,6)(1,5)(3,4)] \pmod{11}$	$[(1,4,5)(6,10)(3,7)] \pmod{11}$
CK_{13}^2	$[(1,2,4)(3,5)(6,9)] \pmod{13}$	the same
	$[(1,6,10)(3,7)(11,12)] \pmod{13}$	
	$[(1,4,10)(3,9)(2,7)] \pmod{13}$	

For $K_{4,4}$ we use the same arguments as for the coloring ϕ_1. □

Corollary 4.13. *$CH|CG^2$ for $G \in \{K_{3a,4b}, K_{4a,4b}\}$.*

Observation. In order to make the proof of Theorem 2.1 easier to read we shall use the following:

$$K_{4m} = K_{4(m-1)} \cup K_{4,4(m-1)} \cup K_4, \quad m \geqslant 1 \tag{4}$$

or

$$K_{4m} = K_{4(m-2)} \cup K_{8,4(m-2)} \cup K_8, \quad m \geqslant 2 \tag{5}$$

and

$$K_{4m+1} = K_{4(m-2)} \cup K_{9,4(m-2)} \cup K_9, \quad m \geqslant 2. \tag{6}$$

4.2. Proof of Theorem 2.1

We shall accomplish the proof of Theorem 2.1 in (i)–(xiii), for each of the 13 graphs having 2 or 4 edges.

(i) Let P_3 be denoted (x, y, z), where the first edge will be of color A. We give the base blocks needed for the colored design ($m \geqslant 2$):

$$CK_{2m+1}^2 = (0, 1+i, 2m-i) \pmod{2m+1}, \ 0 \leqslant i \leqslant m-1.$$

$$CK_{2m}^2 = \{(\infty, 1, 2), (2, 1, \infty)\} \pmod{2m-1}, \ \{(1, 3+i, 5+2i)\}$$

$$\pmod{2m-1}, 0 \leqslant i \leqslant m-3, \ m \geqslant 3. \text{ For } m = 2 \text{ one has:}$$

$$\{(0, 1, 2)(0, \infty, 1)\} \pmod 3. \text{ For } m = 1 \ P_3 \nmid K_2^2.$$

(ii) Denote the graph $2K_2$ by $[(x, y)(u, v)]$, where the first edge is colored A. The base blocks are:

$$CK_{2m+1}^2 = [(0, 1+i)(2m-i, 2m-1-2i)] \pmod{2m+1},\ 0 \leqslant i \leqslant m-1,\ m \geqslant 2.$$
$$\text{For } m = 1, 2K_2 \nmid K_3^2.$$
$$CK_{2m}^2 = \{[(\infty, 1)(2, 3)], [(2, 3)(1, \infty)]\} \pmod{2m-1},$$

$\{[(1+i, 3+2i)(2+i)(4+2i)]\} \pmod{2m-1}, 0 \leqslant i \leqslant m-3,\ m \geqslant 3$ For $m = 2$ one has: $[(0, 1)(3, 2)], [(3, 2)(0, 1)], [(0, 3)(1, 2)], [(1, 2)(0, 3)],\quad [(0, 2)(1, 3)], [(1, 3)(0, 2)]$. For $m = 1$ $2K_2 \nmid K_2^2$.

(iii) For P_5 let K_{4m}, K_{4m+1} be as in (5) and (6), respectively. We use induction on m. For $m = 2$ *and* 3 it was proved in Lemma 4.3. Hence, by the induction hypothesis, and Corollary 4.2 we are done.

(iv) For $K_{1,4}$ there is only one coloring, $AABB$ and the following decompositions cover all cases.

$$CK_{8m+1}^2 = \{(0; 4i+1, 4i+2, 4i+3, 4i+4), (0; 4i+4, 4i+3, 4i+2, 4i+1)\}$$
$$\pmod{8m+1},\ 0 \leqslant i \leqslant m-1$$

CK_{8m+5}^2 same blocks as for $8m+1$ but with mod $8m+5$ and with the additional block

$$(0; 4m+1, 4m+2, 4m+3, 4m+4) \pmod{8m+5}.$$

$CK_{8m}^2 = \{(0; 4i+1, 4i+2, 4i+3, 4i+4), (0; 4i+4, 4i+3, 4i+2, 4i+1)\} \pmod{8m-1}$, $0 \leqslant i \leqslant m-2$,

$$\{(0; 4m-3, 4m-2, 4m-1, \infty), (0; \infty, 4m-1, 4m-2, 4m-3)\}$$
$$\pmod{8m-1}.$$

Notice that for $m = 1$ only the last two blocks occur.

CK_{8m+4}^2same blocks as for $8m$ but mod $8m+3$ and the additional block

$$(0; 4m, 4m+1, 4m+2, 4m+3) \pmod{8m+3}.$$

(v) For $4K_2$ we have only the coloring $AABB$ and the base blocks are:

$$CK_9^2 = \{[(0, 4)(1, 3)(2, 5)(6, 7)], [(6, 7)(2, 5)(1, 3)(0, 4)]\} \pmod 9.$$
$$CK_{8m+1}^2 = \{[(0, 1)(2, 4)(3, 6)(5, 9)], [(5, 9)(3, 6)(2, 4)(0, 1)],$$
$$[(0, 4i+5)(2, 4i+8)(3, 4i+10)(6, 4i+14)],$$
$$[(6, 4i+14)(3, 4i+10)(2, 4i+8)(0, 4i+5)]\} \pmod{8m+1},$$
$$0 \leqslant i \leqslant m-2$$
$$CK_{8m+5}^2 = \{[(0, 1)(2, 4)(3, 6)(5, 9)], [(0, 3)(1, 5)(2, 8)(4, 9)],$$
$$[(0, 6)(2, 7)(3, 5)(8, 9)],$$

$$[(0,\,4i+7)(2,\,4i+10)(3,\,4i+12)(5,\,4i+15)],$$
$$[(5,\,4i+15)(3,\,4i+12)(0,\,4i+7)(2,\,4i+10)]\}\,(\operatorname{mod} 8m+5),$$
$$0 \leqslant i \leqslant m-2, m \geqslant 2.$$

For $m = 1$ only the first three blocks occur.

$$CK^2_{8m} = \{[(\infty,6)(1,2)(3,5)(4,7)],[(4,7)(3,5)(1,2)(\infty,6)],$$
$$[(0,\,4i+4)(1,\,4i+6)(2,\,4i+8)(3,\,4i+10)],[(3,\,4i+10)(2,\,4i+8)$$
$$(0,\,4i+4)(1,\,4i+6)]\}$$
$$(\operatorname{mod} 8m-1), 0 \leqslant i \leqslant m-2,\ m \geqslant 2.$$

For $m = 1$ only the first two blocks occur.

$$CK^2_{8m+4} = \{[(0,4)(2,7)(1,5)(3,8)],\ [(\infty,1)(2,3)(4,6)(5,8)],$$
$$[(5,8)(4,6)(2,3)(\infty,1)],$$
$$[(0,\,4i+6)(1,\,4i+8)(2,\,4i+10)(3,\,4i+12)],$$
$$[(3,\,4i+12)(2,\,4i+10)(1,\,4i+8)(0,\,4i+6)]\}\,(\operatorname{mod} 8m+3), 0 \leqslant i \leqslant m-2.$$

For $m = 1$ only the first three blocks occur.

(vi) For H_5 let K_{4m}, K_{4m+1} be as in (5) and (6), respectively. We use induction on m. For $m = 2$ *and* 3 it was proved in Lemma 4.4. Hence, by the induction hypothesis, Lemma 4.4 and Corollary 4.5 we are done.

(vii) For C_4 by Lemma 4.6 the statement holds for K_4, K_5 and K_9. This follows also for K_8 using (4). Thus we are done for $m = 1,2$. Hence, we can start induction. Since K_{4m} and K_{4m+1} can be written as in (5) and (6) the result is obtained using Corollary 4.7.

(viii) For $K_{1,3} \cup K_2$ let K_{4m}, K_{4m+1} be as in (5) and (6), respectively. We use induction on m. For $m = 2$ *and* 3 it was proved in Lemma 4.8. Hence, by the induction hypothesis, and Lemma 4.8 and Corollary 4.9 we are done.

(ix) For $P_4 \cup K_2$ let K_{4m}, K_{4m+1} be as in (5) and (6), respectively. We use induction on m. For $m = 2$ *and* 3 it was proved in Lemma 4.10. Hence, by the induction hypothesis, and Lemma 4.10 and Corollary 4.11 we are done.

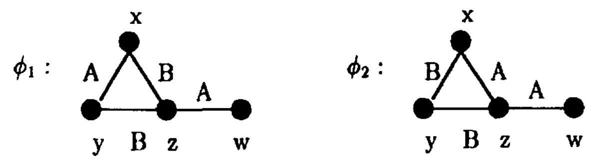
(x) For $P_3 \cup 2K_2$ let K_{4m}, K_{4m+1} be as in (5) and (6), respectively. We use induction on m. For $m = 2$ *and* 3 it was proved in Lemma 4.12. Hence, by the induction hypothesis, and Lemma 4.12 and Corollary 4.13 we are done.

(xi) Let $2P_3$ be denoted $[(x,y,z)(u,v,w)]$. We give the base blocks for the two possible colorings $\phi_1 : AA, BB$ and $\phi_2 : AB, BA$.

$$CK^2_{4m+1} = \{[(0+i,\,4m-i,\,2m+i)(1+i,\,4m-1-i,\,2m+1+i)]$$
$$(\operatorname{mod} 4m+1), 0 \leqslant i \leqslant m-2,\ m \geqslant 2\}, \{[(m-1,\,3m+1,\,3m-1)$$
$$(1,\,4m,\,2m+1)](\operatorname{mod} 4m+1)\}.$$

For K_{4m} we take the same base blocks but replace the $2m$ in the first and in the last block by ∞, reduce the blocks and do the shifting $\bmod 4m-1$.

(xii) Let H be the triangle with a pendant edge. Denote by $(x, y, z; w)$ the triangle (x, y, z) with attached edge (z, w). There are two possible colorings of H namely, $\phi_1 : (z, w)$ colored like (x, y) and $\phi_2 : (z, w)$ colored like (x, z), thus



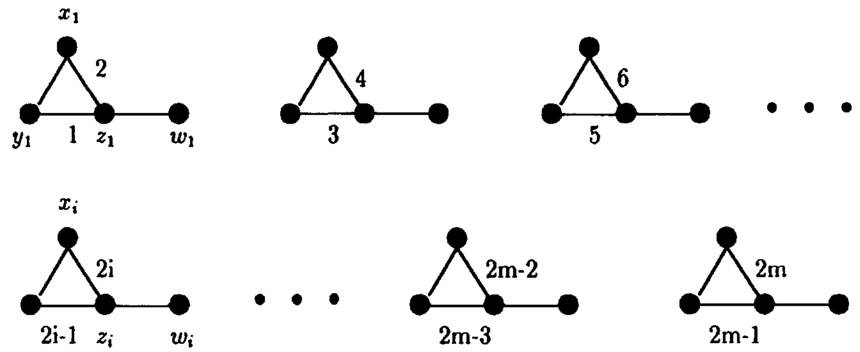
We shall give in (a)–(d) below the colored blocks for ϕ_1 and ϕ_2 for the cases $n \in \{4m, 4m+1\}$.

(a) For ϕ_1 we have $CK^2_{4m+1} = (5+2i, 2-2i, 3; 5+2i+t) (\bmod 4m+1), 0 \leqslant i \leqslant m-1$, where t is defined by

$$t = \begin{cases} -3 & \text{for } i \text{ odd,} \\ 2 & \text{for } i \text{ even if } i < m-1, \\ -1 & \text{for } i = m-1 \text{ and even.} \end{cases}$$

The above decomposition holds for $m \geqslant 1$. To check the correctness of the colored decomposition one has to get convinced not only that each difference $1, 2, \ldots, 2m$ in absolute value occurs exactly twice but also that the occurence is once in each color.

We shall show in detail how twice $2m$ differences have been distributed on m copies of H providing the m base blocks. For simplicity let m be even, $m = 2k$. Place first the differences $1, 2, \ldots, 4k$ by consecutive values on the edges $(y,z), (x,z)$ of the triangle in H adjacent to the attached edge as follows:



and choose all of them to have color B. Now put on (x_i, y_i) the difference $1-4i$ if the sum of the other two is smaller than $2m$ and $4m-4i+2$ otherwise. In this way we get m values for (x_i, y_i) all different and each element of $\{1,2,\ldots,2m\}$ in absolute value. The other m values shall be assigned to (z_i, w_i) and are the values

$\{4,1,8,5,12,\ldots,4k,4k-3\}$ taken in that order. The last $2m$ differences are all chosen color A.

(b) For ϕ_2 we have $CK^2_{4m+1} = (-4i, 1-2i, 3; 4\lceil i/2\rceil + t)(\bmod 4m+1), 0 \leqslant i \leqslant m-1$, where t is defined by,

$$t = \begin{cases} 0 \text{ for } i \text{ odd,} \\ 7 \text{ for } i \text{ even and } < m-1, \\ 4 \text{ for } i = m-1 \text{ and even.} \end{cases}$$

This holds for $m \geqslant 1$.

The colored base blocks above can be obtained in a way similar to that in case (a). In this case the consecutive pairs from $\{1,2,\ldots,4k\}$ are placed on the edges $(x,y),(y,z)$.

(c) For ϕ_1 we have $CK^2_{4m} = (6+2i, 1-2i, 3; 6+2i+t)(\bmod 4m-1)$ and in addition the block $(\infty,2,3;4)(\bmod 4m-1)$ for m odd, and the blocks $(\infty,2,3;2m+2)(\bmod 4m-1)$, $(2m+2, 5-2m, 3; 4)(\bmod 4m-1)$ for m even. In the first case $0 \leqslant i \leqslant m-2$ while in the last one $0 \leqslant i \leqslant m-3$, where, t is defined to be 1 for i even and -2 for i odd. The above holds for $m \geqslant 1$.

Concerning the obtaining of the above blocks the method is similar to that used in cases (a) and (b). Here the difference 1 is used twice in the additional blocks. The consecutive pairs of differences from $\{2,3,\ldots,2m-1\}$, except those used in the additional blocks are assigned to the edges $(x,z)(y,z)$ with the same color.

(d) For ϕ_2 we have $CK^2_{4m} = (8+4i, 6+2i, 3; 6+2i+t)(\bmod 4m-1)$ and in addition the block $(\infty,2,3;4)(\bmod 4m-1)$ for m odd, and the blocks $(\infty, 2m+4, 3; 5)(\bmod 4m-1)$, $(2m+2, 2, 3; 4)(\bmod 4m-1)$ for m even. In the first case $0 \leqslant i \leqslant m-2$ while in the last one $0 \leqslant i \leqslant m-3$.

The parameter t is defined to be 1 for i even and -2 for i odd.

In this case the pair of differences starting with 2,3 are assigned to the edges $(x,y)(y,z)$ with the same color.

(xiii) If $H = K_3 \cup K_2$ then there is only one possible coloring. Denote H by $(x,y,z)(u,v)$. We have

(a) $CK^2_{4m+1} = (5+2i, 2-2i, 3)(3+s, 5+2i+t+s)(\bmod 4m+1)$,

where i and t are as in xii(a), while s is 2 for odd i and -3 for even i. This is valid for $m \geqslant 2$. Notice that the triangle in the block is as in xii(a) and the edge is obtained by adding s to the labels of (z,w). The definition of s insures that $z+s$ and $w+s$ differ modulo $4m+1$ from each of x,y,z.

(b) $CK^2_{4m} = (6+2i, 1-2i, 3)(3+s, 6+2i+t+s)(\bmod 4m-1)$,

i and t are as in xii(c) for the standard blocks with $s=1$ for i even and $s=3$ for i odd, while the additional blocks are:

$(\infty,2,3)(4,5)$, for m odd and $(\infty,2,3)(4,2m+3), (2m+2, 5-2m, 3)(4,5)$ for m even, each mod $(4m-1)$.

This completes the case $\lambda = 2$ with all graphs having two or four edges. □

4.3. Preliminary results in the case $\lambda = 3$

4.3.1. Building blocks for P_4

We denote P_4 by (x,y,z,w) and there is only one possible coloring and we shall denote it, ABC.

Lemma 4.14. *$CP_4|CG^3$ for $G \in \{K_{2,2}, K_{2,3}, K_{3,3}, K_i,\ i = 4,5,6,7\}$.*

Proof. The desired decompositions are:

$CK^3_{2,2}$	$(0,a,1,b)$	$(a,1,b,0)$	
	$(1,b,0,a)$	$(b,0,a,1)$	
$CK^3_{2,3}$	$(0,a,1,b)$	$(b,0,a,2)$	
	$(1,b,0,a)$	$(1,a,2,b)$	
	$(a,2,b,0)$	$(2,b,1,a)$	
$CK^3_{3,3}$	$(0,a,1,b)$	$(0,b,1,c)$	$(0,c,1,a)$
	$(1,b,2,c)$	$(1,c,2,a)$	$(1,a,2,b)$
	$(2,c,0,a)$	$(2,a,0,b)$	$(2,b,0,c)$
CK^3_4	$(\infty,0,1,2,)\,(\text{mod }3)$	$(0,1,\infty,2)\,(\text{mod }3)$	
CK^3_5	$(0,1,2,3)\,(\text{mod }5)$	$(0,2,4,1)\,(\text{mod }5)$	
CK^3_6	$(\infty,1,2,4)\,(\text{mod }5)$	$(2,1,\infty,0)\,(\text{mod }5)\,(3,0,2,1)\,(\text{mod }5)$	
CK^3_7	$(0,1,2,3)\,(\text{mod }7)$	$(0,2,4,6)\,(\text{mod }7)$	$(0,3,6,2)\,(\text{mod }7)$

Corollary 4.15. *$CP_4|CK^3_{m,n}$ for all $m,n \geqslant 2$.*

4.3.2. Building blocks for $H = P_3 \cup K_2$

We denote the graph H by $[(x,y,z)(u,v)]$ and the colors are A,B,C in that order.

Lemma 4.16. *$CH|CG^3$, for $G \in \{K_{2,3}, K_{3,3}, K_6, K_8\}$.*

Proof. The required colored designs are:

$CK^3_{2,3}$	$[(0,a,1)(b,2)]$	$[(0,b,1)(a,2)]$
	$[(1,a,2)(b,0)]$	$[(1,b,2)(a,0)]$
	$[(2,a,0)(b,1)]$	$[(2,b,0)(a,1)]$
$CK^3_{3,3}$	$[(0,a,1)(c,2)]$	$[(1,a,2)(c,0)]$
	$[(2,a,0)(c,1)]$	$[(0,b,1)(a,2)]$
	$[(1,b,2)(a,0)]$	$[(2,b,0)(a,1)]$
	$[(0,c,1)(b,2)]$	$[(1,c,2)(b,0)]$
	$[(2,c,0)(b,1)]$	

CK_6^3	$[(\infty,1,2)(3,4)]$	$[(3,1,\infty)(2,4)]\,(\text{mod}\,5)$
	$[(1,2,4)(\infty,0)]\,(\text{mod}\,5)$	
CK_8^3	$[(\infty,1,2)(3,4)]$	$[(3,1,\infty)(2,4)]\,(\text{mod}\,7)$
	$[(1,2,4)(\infty,0)]$	$[(1,4,0)(3,6)]\,(\text{mod}\,7)$

Corollary 4.17. *$CH|CG^3$, for $G \in \{K_{2a,3b}, K_{3a,4b}\}$.*

4.3.3. Building blocks for $K_{1,3}$ and $3K_2$

For each of the graphs $K_{1,3}$ and $3K_2$, there exists only one coloring and it is permutable. Hence, in virtue of Corollary 1.2 we have:

Theorem 4.18. *If $H|G$ then, $CH|CG^3$ for $H \in \{K_{1,3}, 3K_2\}$.*

The following lemma is simple. We omit the proof.

Lemma 4.19. *$C(3K_2)|C(4K_2)^3$.*

Lemma 4.20. *$C(3K_2)|CG^3$, for $G \in \{K_8, K_{11}\}$.*

Proof. For K_8 observe that $4K_2|K_8$. Then we apply Theorem 4.18 together with Lemma 4.19.

For, K_{11} one has the decomposition:

$$CK_{11}^3 = \{[(1,2)(3,4)(5,6)], [(1,3)(2,4)(5,7)], [(1,4)(2,5)(3,6)], [(1,5)(2,6)(3,7)], [(1,6)(2,7)(3,8)]\}\,(\text{mod}\,11). \qquad \square$$

Lemma 4.21. *$CK_{1,3}|CG^3$ for $G \in \{K_5, K_{1,3s+1}\}$.*

Proof. For K_5, one has

$$CK_5^3 = \{(0;1,4,2), (0;2,3,1)\}\,(\text{mod}\,5).$$

Let the star $K_{1,4}$ be $(P;a,b,c,d)$. Then the decomposition of $CK_{1,4}^3$ is

$$(P;a,b,c), (P;d,a,b), (P;c,d,a), (P;b,c,d).$$

4.4. Proof of Theorem 2.2

We shall accomplish the proof of the theorem in (i)–(v) below, for each of the five graphs having three edges.

(i) To prove that $CP_4|CK_n^3$, for all $n \geqslant 4$ we use induction on n. The cases $4 \leqslant n \leqslant 7$ were proved in Lemma 4.14. Let,

$$K_n = K_4 \cup K_{4,n-4} \cup K_{n-4}.$$

By the induction hypothesis, Lemma 4.14 and Corollary 4.15, we are done.

(ii) We prove for $H = P_3 \cup K_2$ that $CH|CK_n^3$, for all $n \geqslant 5$. For n odd it is well known [6] that K_n has a Hamilton cycle decomposition. Let C_n be such a Hamilton cycle with vertices labeled $(0,1,2,\dots,n-1,0)$. A colored decomposition of C_n^3 is given by $[(0,1,2)(3,4)] \pmod n$. This implies the colored decomposition of K_n^3 for n odd.

For n even we use induction on n. For $n \in \{6,8\}$ a decomposition was given in Lemma 4.16. For $n = 10$, we write $K_{10} = K_5 \cup K_{5,5} \cup K_5$ and apply the result for K_5 together with Lemma 4.16 and the fact that $K_{2,3}|K_{5,5}$. For $n \geqslant 12$ let,

$$K_n = K_6 \cup K_{6,n-6} \cup K_{n-6}.$$

By the induction hypothesis, Lemma 4.16 and Corollary 4.17 we are done.

(iii) In the case of K_3 it was noted in [5] that this case was settled in [8,9] using the concept of perpendicular arrays of strength 3.

(iv) It is well-known [10] that $K_{1,3}|K_n$ if $n \equiv 0, 1 \pmod 3, n \neq 3,4$. In view of Theorem 4.18 and since the design does not exist when $n \in \{3,4\}$, we have only to prove the statement in the case $n \equiv 2 \pmod 3$. Since $K_{3m+2} = K_{3m+1} \cup K_{1,3m+1}$ and $K_{1,3}|K_{3m+1}$, by Lemma 4.21 and the fact that $K_{1,3m+1} = (m-1)K_{1,3} \cup K_{1,4}$ we are done.

(v) For $3K_2$ it is known [2,12] that $3K_2|K_n$ for $n \equiv 0, 1 \pmod 3, n \geqslant 6$. Hence in virtue of Theorem 4.18 we need to solve the colored design problem only for $n = 3m+2, n \geqslant 6$. For, $m = 2$ *and* 3 a decomposition was given in Lemma 4.20. Let

$$K_{3m+2} = K_8 \cup K_{8,3(m-2)} \cup K_{3(m-2)}.$$

Hence, by the $3K_2$ decomposition of $K_{3(m-2)}$, and Lemma 4.19, we are done.

This completes the proof of the case $\lambda = 3$ and all the graphs having three edges. □

4.5. Final remark

In this paper we have limited our attention to small graphs H (at most four edges), and uniform colorings to a small number of colors ($\lambda = 2,3$). We did so since this allows us to present a complete and constructive solution to the existence problem of the corresponding designs. In a forthcoming paper [4] we consider large graphs H and a large number of colors. In particular, we have some general results concerning the path P_{2t+1}, and the graphs $2tK_2, tP_3$ and $K_{1,2t}$ for all $t \geqslant 1$, namely,

Theorem A. *If ϕ is a symmetric coloring, then $C_\phi K_{2t+1}|CK_n^2$ for $n \equiv 0, 1 \pmod{2t}$, $n > 2t$.*

Theorem B.
(a) $C(2tK_2)|CK_n^2$ *for* $n \equiv 0, 1 \pmod{2t}$, $n \geqslant 4t$,
(b) $C(tP_3)|CK_n^2$ *for* $n \equiv 0, 1 \pmod{4t}$, $n \geqslant 3t$,
(c) $C(K_{1,2t})|CK_n^2$ *for* $n \equiv 0, 1 \pmod{2t}$, $n > 2t$.

Another direction of fruitful investigations looks like considering the nonuniform colorings of the small graphs presented in this paper.

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